1. Heat conduction in a rod

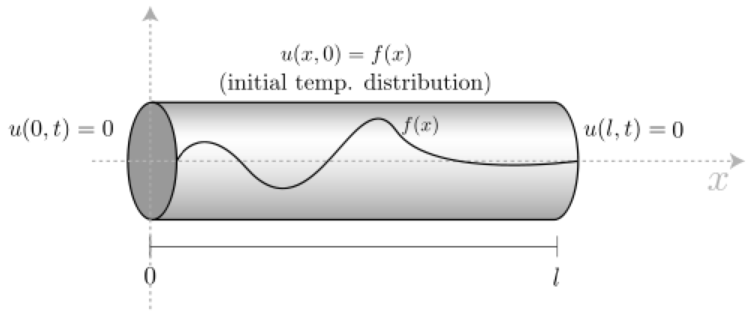

Figure 1: Idealized physical setting for heat conduction in a rod with homogeneous boundary conditions.

The temperature in the rod satisfies the heat conduction equation which has the form

$$\begin{aligned} \alpha^2 u_{xx} &= u_t, \qquad 0 < x < l,\ t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le l \text{ (initial condition)}, \\ u(0,t) &= u(l,t) = 0, \quad t > 0 \text{ (boundary conditions)}. \end{aligned} \tag{1}$$

where α^2 is a constant known as the thermal diffusivity.

(a) The temperature at a point on the rod depends on two variables. What are they?

(b) The function u is separable $u(x,t) = X(x)T(t)$. Substitute for $u(x,t)$ in the boundary condition at $x = 0$ and $x = l$, find boundary conditions for $X(x)$.

(c) Show that X and T satisfy

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$$

where λ is the separation constant.

(d) Notice that we obtain the following two ordinary differential equations for $X(x)$ and $T(t)$:

$$X'' + \lambda X = 0, \tag{2}$$

$$T' + \alpha^2 \lambda T = 0. \tag{3}$$

First, solve the the eigenvalue problem given by (2) and by using the eigenvalues solve (3).

(e) Note that if $X_n(x)$ and $T_n(t)$ are the eigenfunctions corresponding to eigenvalues λ_n, then $u_n(x,t) = X_n(x)T_n(t)$ is a solution for (1). Therefore, we can assume that

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t). \tag{4}$$

Based on that, write $u(x,t)$ as in (4).

(f) Determine the coefficients c_n (Hint: use the initial condition given in (1) and Fourier sine series).

Congratulations, now you know how to solve a heat conduction equation☺

Let's practice to improve your new skill!

2. Separation of variables

Find the solution of the heat conduction problem

$$u_{xx} = 4u_t,\ \ 0 < x < 2,\ \ t > 0;$$
$$u(0,t) = u(2,t) = 0,\ \ t > 0;$$
$$u(x,0) = 12\sin(\frac{9\pi x}{2}) - 7\sin(\frac{\pi x}{2}).$$

3. Heat Conduction Equation

Suppose that we have an insulated wire of length 1, such that the ends of the wire are embedded in ice (temperature 0). Let $\alpha^2 = 0.003$. Then suppose that initial heat distribution is $u(x,0) = 50x(1-x)$. Find the temperature function $u(x,t)$. After formulating $u(x,t)$, check that $u(x,t)$ can be plotted as in the Figure 2 (Hint: Write $f(x)$ as sine series for $0 < x < 1$).

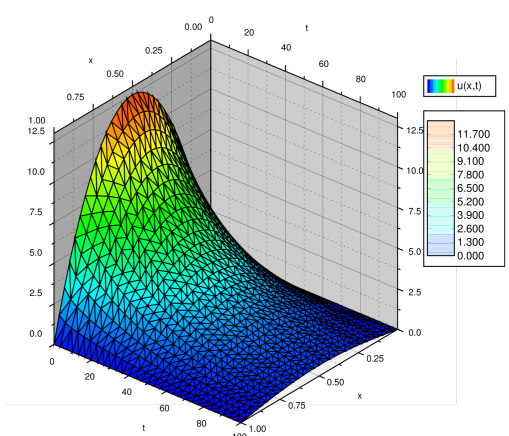

Figure 2: Plot of the temperature of the wire, $u(x,t)$, at position x at time t.

4. Practice more and check your understanding!

One of the best way to check how well you understand a topic is trying to write a question about it. Can you write a real-life problem which involves separation of variables and Fourier series?

Thanks for hosting me ! Feel free to contact me if you have questions.

5. Contact Information

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You can find solutions of Worksheet 11, 12 at `http://rtsl-edge.cs.illinois.edu/nerimanno/?teaching`.

Home of the Fourier transform family ☺☺

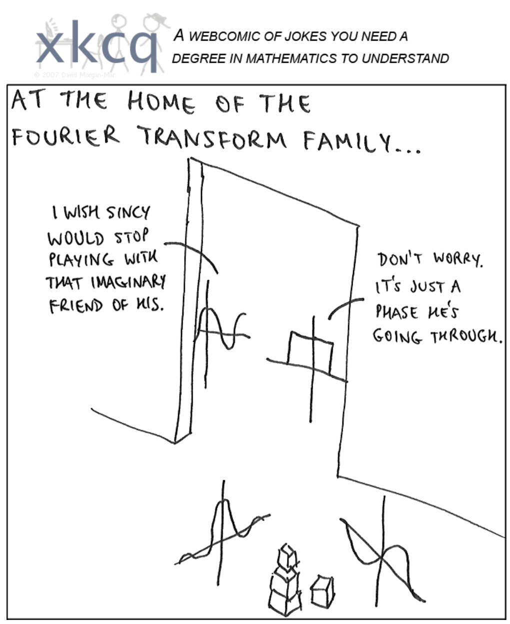

Figure 3: Have you ever wondered if a rectangular signal is physically possible? Check out `https://en.wikipedia.org/wiki/Sinc_function`, you will meet Sincy.