# Worksheet 11 – Solutions

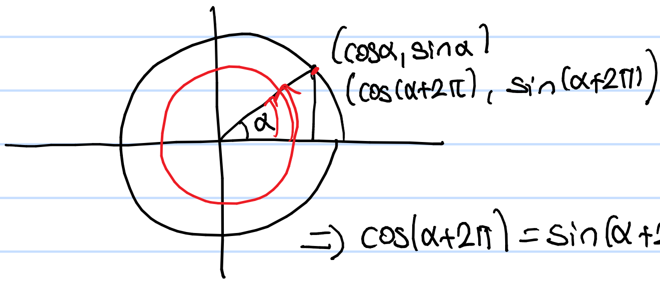
1. Let $f$ be a periodic function with period $T$, i.e.

$$f(x+T) = f(x)$$

whenever $x$ and $x+T$ is in the domain $f$.

We can show that $f(\alpha x)$ has period $\frac{T}{\alpha}$ (Exercise).

Notice that sine and cosine are periodic functions with period $2\pi$.




$\Rightarrow \cos(\alpha+2\pi) = \sin(\alpha+2\pi)$

a) $\sin\left(\frac{m\pi x}{L}\right)$ is periodic with period $\frac{2\pi}{\frac{m\pi}{L}} = \frac{2L}{m}$

b) $\cos\left(\frac{m\pi x}{L}\right)$ is periodic with period $\frac{2\cancel{\pi}}{\frac{m\cancel{\pi}}{L}} = \frac{2L}{m}$

c) $\tan x$ is a periodic function with period $\pi$, then $\tan(3\pi x)$ is periodic with period $\frac{\pi}{3\pi} = \frac{1}{3}$

(d) $f(x) = x^2 + 3x$, it is not a periodic function.

(e) $f(x) = \begin{cases} (-1)^n, & 2n-1 \le x < 2n \\ 1, & 2n \le x < 2n+1; \end{cases}$ $n = 0, \mp 1, \mp 2, \dots$

Let $x \in [2n-1, 2n)$, $n = 0, \mp 1, \mp 2, \dots$

$f(x) = (-1)^n$

$f(x+1)$: $2n \le x+1 < 2n+1 \Rightarrow f(x+1) = 1 \neq (-1)^n$ for odd n

$f(x+2)$: $\underbrace{2n+1}_{2(n+1)-1} \le x+2 < 2n+2 \Rightarrow f(x+2) = (-1)^{n+1} = -(-1)^n \neq (-1)^n$

$f(x+3)$: $2n+2 \le x+3 < 2n+3 \Rightarrow f(x+3) = 1 \neq (-1)^n$ for odd n

$f(x+4)$: $2n+3 \le x+4 < 2n+4 \Rightarrow f(x) = (-1)^{n+2} = (-1)^n (-1)^2 = (-1)^n$

Then period is 4.

Another way to approach this part: $\Rightarrow$

$$\left.\begin{aligned} f(x) &= (-1)^n, \quad 2n-1 \le x < 2n \\ f(x) &= 1, \quad 2n \le x < 2n+2 \end{aligned}\right\}$$ We will find the periods of this parts separately.

- $f(x) = (-1)^n, \quad 2n-1 \le x < 2n$, domain $D = \bigcup_{n=1}^{\infty} [2n-1, 2n)$

If $x \in D$, then $x + 2k \in D$ for $k = 0, 1, 2, \ldots$

$$f(x+2) = 1 \ne (-1)^n \text{ for odd } n$$
$$f(x+4) = \underbrace{(-1)^{n+2}}_{\text{as we did before}} = (-1)^n$$

$\Rightarrow$ Period for this part is 4.

- $f(x) = 1, \quad 2n \le x < 2n+2$, domain $D = \bigcup_{n=1}^{\infty} [2n, 2n+2)$

If $x \in D$, then $x + (2k) \in D$ for $k = 1, 2, 3, \ldots$

$$\left.\begin{aligned} f(x) &= 1 \\ f(x+2) &= \underline{\underline{1}} \end{aligned}\right\} \Rightarrow \text{Period is } 2$$

It's a constant function.

Note that any multiple of a period is also period

$\Rightarrow \operatorname{lcm}(2, 4) = 4$ is the fundamental period of $f$.

**Problem 2**: We will show that the set

$\{\sin(mx), \cos(nx), \; m,n \in \mathbb{Z}^+\}$ forms a

mutually orthogonal set of real-valued functions.

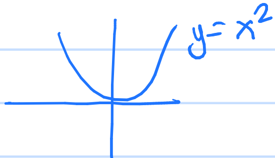
After solving this problem, you will be able to determine Fourier coefficients which are given by Euler-Fourier formula.

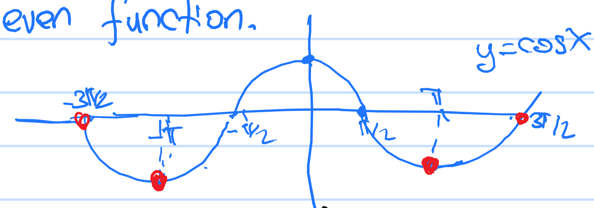
- We will use the properties of even and odd functions

If $f(x) = f(-x)$ for all $x, -x \in D(f)$ (→ domain), then

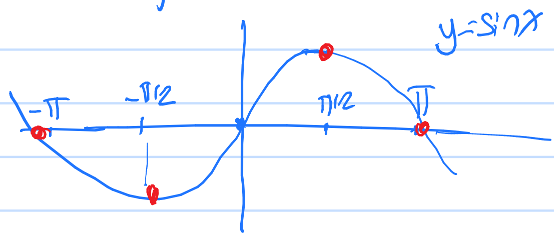
$f$ is said to be an even function.

Ex:




If $f(-x) = -f(x)$ for all $x, -x \in D(f)$, then
$f$ is said to be an odd function.

## If check the following:

- Any linear combination of even functions is is even.
- Any linear combination of odd functions is odd.
- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and odd function is odd.

integral of even-odd functions

If $f$ is an even function on the interval $-L \leq x \leq L$, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

If $f$ is an odd function on the interval $-L \leq x \leq L$, then

$$\int_{-L}^{L} f(x)\,dx = 0$$

We will need some trigonometric identities:

i) $\cos(\alpha \pm \beta) = \cos(\alpha).\cos(\beta) \mp \sin(\alpha).\sin(\beta)$

ii) $\sin(\alpha \pm \beta) = \sin(\alpha).\cos(\beta) \pm \cos(\alpha).\sin(\beta)$

iii) $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ (You can derive it from (i))

iv) $\cos^2\theta = \dfrac{1 + \cos(2\theta)}{2}$

v) $\sin^2\theta = \dfrac{1 - \cos(2\theta)}{2}$

follows from (iii)

a) $\cos(\alpha+\beta) = \cos(\alpha).\cos(\beta) - \cancel{\sin(\alpha).\sin(\beta)}$

$\cos(\alpha-\beta) = \cos(\alpha).\cos(\beta) + \cancel{\sin(\alpha).\sin(\beta)}$ +

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos(\alpha).\cos(\beta)$$

$$\Rightarrow \cos(\alpha).\cos(\beta) = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$$

$$\Rightarrow \int_{-\pi}^{\pi} \cos(mx).\cos(nx) = \frac{1}{2}\int_{-\pi}^{\pi} \underbrace{\cos(m+n)x + \cos(m-n)x}_{\text{even function}}$$

$$= \frac{1}{2}.2\int_{0}^{\pi} \cos(m+n)x + \cos(m-n)x . dx$$

$$= \int_{0}^{\pi} \cos(m+n)x + \cos(m-n)x \, dx$$

$\Rightarrow$

$$= \frac{1}{m+n} \cdot \sin(m+n)x + \frac{1}{m-n} \sin(m-n)x \Big|_0^{\pi}$$

$$= \frac{1}{m+n} \cdot \cancel{\sin(m+n)\pi}^{\,0} + \frac{1}{m-n} \cancel{\sin(m-n)\pi}^{\,0}$$

$$- \frac{1}{m+n} \cancel{\sin 0}^{\,0} - \frac{1}{m-n} \cancel{\sin(0)}^{\,0} = 0$$

$\left( \sin(\pi k) = 0, \; k = 0, 1, 2, \ldots \right)$

- $$\int_{-\pi}^{\pi} \sin(mx) \cdot \sin(nx) \cdot dx$$

$$\sin(mx) \cdot \sin(nx) = \frac{\cos(m-n)x - \cos(m+n)x}{2}$$ (As we did for $\cos(mx) \cdot \cos(nx)$)

$$\Rightarrow \int_{-\pi}^{\pi} \underbrace{\frac{\cos(m-n)x - \cos(m+n)x}{2}}_{\text{even function}} \cdot dx = \int_0^{\pi} \cos(m-n)x - \cos(m+n)x \cdot dx$$

$$= \frac{1}{m-n} \sin(m-n)x - \frac{1}{m+n} \sin(m+n)x \Big|_0^{\pi} = 0$$

(b) $\int_{-\pi}^{\pi} \cos(mx)\cdot\sin(nx)\cdot dx$

$$\sin(\alpha+\beta) = \cancel{\sin\alpha\cdot\cos\beta} + \cos\alpha\cdot\sin\beta$$
$$- \quad \sin(\alpha-\beta) = \cancel{\sin\alpha\cdot\cos\beta} - \cos\alpha\cdot\sin\beta$$

$$\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\cos\alpha\cdot\sin\beta$$

$$\Rightarrow \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} = \cos\alpha\cdot\sin\beta$$

- $$\int_{-\pi}^{\pi} \cos(mx)\cdot\sin(nx)\,dx = \int_{-\pi}^{\pi} \underbrace{\frac{\sin(m+n)x+\sin(m-n)x}{2}}_{\text{odd function}}\,dx = 0$$

- $$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2} \quad (iv)$$

$\rightarrow$ even function

$$= \int_{0}^{\pi} 1+\cos(2nx)$$

$$= x + \frac{\sin(2nx)}{2n}\Bigg|_0^{\pi}$$

$$= \pi + 0 - 0 = \pi$$

- $$\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \int_{-\pi}^{\pi} \underbrace{\frac{1-\cos(2nx)}{2}}_{\text{even function}}\,dx$$

$$= \int_{0}^{\pi} (1-\cos(2nx))\,dx$$

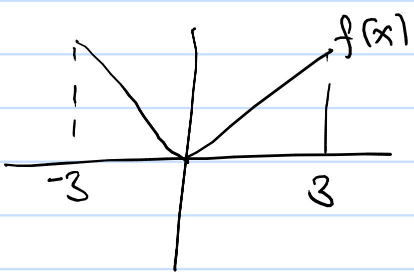
$$= x - \frac{\sin(2nx)}{2n}\Big|_{0}^{\pi} = \pi - 0 - (0-0) = \pi$$

**Problem 4:**

$$f(x) = \begin{cases} -x, & -3 \le x < 0 \\ x, & 0 \le x < 3; \end{cases}$$

$f(x+6) = f(x)$. Determine the Fourier coefficients.

**Solution:** $L = \dfrac{\text{period of } f}{2} = \dfrac{6}{2} = 3$




$\Rightarrow f(x)$ is an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{3}\right) + b_n \cdot \sin\left(\frac{n\pi x}{3}\right)$$

$3 \rightarrow L$

$$a_n = \frac{1}{3} \int_{-3}^{3} \underbrace{\underbrace{f(x)}_{\text{even}} \cdot \underbrace{\cos\left(\frac{n\pi x}{3}\right)}_{\text{even}}}_{\text{even function}} \cdot dx$$ (Euler-formula)

$$= \frac{2}{3} \int_{0}^{3} x \cdot \cos\left(\frac{n\pi x}{3}\right) \cdot dx$$

$\downarrow$ integration by parts

$$= \frac{2}{3} \left( \frac{3}{n\pi} \sin\left(\frac{n\pi x}{3}\right) \cdot x + \left(\frac{3}{n\pi}\right)^2 \cdot \cos\left(\frac{n\pi x}{3}\right) \right) \Bigg|_0^3$$

$$= \frac{2}{3} \left( \frac{3}{n\pi} \cdot \underbrace{\cancel{\sin(n\pi)}}_{0} \cdot 3 + \left(\frac{3}{n\pi}\right)^2 \underbrace{\cos(n\pi)}_{\downarrow} - 0 - \left(\frac{3}{n\pi}\right)^2 \cdot \underbrace{\cos(0)}_{1} \right)$$

$\downarrow$ depends on if n is odd or even

$$= \frac{2}{\cancel{3}} \cdot \frac{\cancel{9}^{3}}{n^2 \pi^2} \cdot (\cos(n\pi) - 1)$$

$$a_n = \begin{cases} \dfrac{-12}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{3}\int_{-3}^{3} \underbrace{\underbrace{f(x)}_{\text{even}} \cdot \underbrace{\sin\left(\frac{n\pi x}{3}\right)}_{\text{odd}}}_{\text{odd}} \cdot dx = 0$$

$$a_0 = \frac{1}{3}\int_{-3}^{3} \underbrace{f(x)}_{\text{even}}\, dx = \frac{2}{3}\int_{0}^{3} x\, dx = \frac{\cancel{2}}{3} \cdot \frac{x^2}{\cancel{2}}\Bigg|_{0}^{3}$$

$$= \frac{x^2}{3}\Bigg|_{0}^{3} = 3$$

$$\Rightarrow f(x) = \frac{3}{2} - \frac{12}{\pi^2} \sum_{\substack{n \text{ is} \\ \text{odd}}} \frac{\cos\left(\frac{n\pi x}{3}\right)}{n^2}$$

$$f(x) = \frac{3}{2} - \frac{12}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{(2k-1)\pi x}{3}\right)}{(2k-1)^2}$$

**Problem 4**: Find the formal solution of the initial value problem

$$y'' + w^2 y = f(t), \quad y(0) = 1, \quad y'(0) = 0$$

$f$ is periodic with period 2 and

$$f(t) = \begin{cases} 1-t, & 0 \le t < 1; \\ -1+t, & 1 \le t < 2 \end{cases}$$

**Solution**: $L = \frac{\text{period}}{2} = 1$

$$y'' + w^2 y = f(t) \qquad y = y_c + y_p$$

**Step 1**: $y_c = ?$

$$r^2 + w^2 = 0 \Rightarrow r = \mp wi$$

$$\Rightarrow y_c = c_1 \cos(wx) + c_2 \sin(wx)$$

**Step 2**: In order to find $y_p$, first we need to write fourier series of $f(t)$

$$f(t) = \begin{cases} 1-t, & 0 \le t < 1 \\ -1+t, & 1 \le t < 2 \end{cases} \qquad f(t+2) = f(t)$$

Please see Problem 27 in page 606:

If $f$ is an integrable periodic function with period $T$ such that $0 \le a \le T$, then

$$\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx$$

$$\int_{-1}^{1} f(t)\,dt = \int_0^2 f(t)\,dt$$

$$= \int_0^1 (1-t)\,dt + \int_1^2 (-1+t)\,dt$$

$$= t - \frac{t^2}{2}\Big|_0^1 + \left(-t + \frac{t^2}{2}\right)\Big|_1^2$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Period 2

$$a_n = \int_0^2 f(t).\cos(n\pi t).dt$$

$$= \int_0^1 (1-t)\cos(n\pi t)dt + \int_1^2 (-1+t)\cos(n\pi t)dt$$

$$= \frac{1-\cos(\pi n)}{\pi^2 n^2} = \begin{cases} \frac{4}{\pi^2 n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \int_0^2 f(t)\sin(n\pi t).dt = 0$$

$$\Rightarrow f(t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi t}{(2n-1)^2}$$

$$y_p = \sum_n \left( A_n.\cos(n\pi t) + B_n \sin(n\pi t) \right)$$

Step 3 $\quad y = \underbrace{y_c}_{\text{homogenous solution}} + y_p$

We know that

AND

$$y_c'' + w^2 y_c = 0$$

$$y_p'' + w^2 y_p = f(t) \quad (*) \Rightarrow$$

$$y_p'' = \sum_n -n^2\pi^2 A_n \cos(n\pi t) - n^2\pi^2 B_n \sin(n\pi t)$$

$$\Rightarrow y_p'' + \omega^2 y_p = (\omega^2 - n^2\pi^2) \sum_n A_n \cos(n\pi t) + B_n \sin(n\pi t)$$

$$y_p'' + \omega^2 y_p = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi t}{(2n-1)^2} \quad f(t)$$

$$= (\omega^2 - n^2\pi^2) \sum_n A_n \cdot \cos(n\pi t) + B_n \sin(n\pi t)$$

$$= \frac{1}{2} + \frac{4}{\pi^2} \sum_{n \text{ is odd}}^{\infty} \frac{\cos(n\pi t)}{n^2}$$

We can let $B_n = 0, \; n = 0, 1 \ldots$

$A_n = 0$, for $n = 2, 4, 6, \ldots$

$$A_0 \cdot \omega^2 = \frac{1}{2} \Rightarrow A_0 = \frac{1}{2\omega^2}$$

$$A_n \cdot (\omega^2 - n^2\pi^2) = \frac{4}{\pi n^2} \Rightarrow A_n = \frac{4}{\pi n^2(\omega^2 - n^2\pi^2)}, \; n \text{ is odd}$$

n is odd

$\Rightarrow$

$$y_p = \frac{1}{2\omega^2} + \frac{4}{\pi^2} \sum_{\substack{\text{odd}\\ n}} \frac{\cos(n\pi t)}{n^2(\omega^2 - \pi^2 n^2)}$$

$$y = y_c + y_p$$

$$y = c_1 \cdot \cos(\omega x) + c_2 \cdot \sin(\omega x) + y_p$$

## Step 4: Check boundar conditions

$$y(0) = c_1 + \frac{1}{2\omega^2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2\left[\omega^2 - (2n-1)^2\pi^2\right]} = 1$$

$$\Rightarrow \quad c_1 = \left(1 - \frac{1}{2\omega^2}\right) - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2\left[\omega^2 - (2n-1)^2\pi^2\right]}$$

$$y(x) = c_1 \cdot \cos(\omega x) + c_2 \sin(\omega x) + \frac{1}{2\omega^2} + \sum_{\substack{\text{odd}\\ n}} A_n \cdot \cos(n\pi x)$$

$$y'(x) = -c_1 \cdot \omega \cdot \sin(\omega x) + c_2 \cdot \omega \cdot \cos(\omega x) + \sum_{\substack{\text{odd}\\ n}} -(n\pi) A_n \cdot \sin(n\pi x)$$

$$\Rightarrow y'(0) = c_2 \cdot \omega = 0$$

$$\Rightarrow \boxed{c_2 = 0} \quad \Rightarrow$$

$$y(x) = C_1 \cdot \cos(\omega x) + A_0 + \sum A_n \cdot \cos(n\pi t)$$

$$y(x) = \frac{1}{2w^2} + \left(1 - \frac{1}{2w^2}\right)\cos(wx) + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x - \cos wx}{(2n-1)^2\left[w^2 - (2n-1)^2\pi^2\right]}$$