Math 199	Fourier Series (Section 10.2, 10.3)	November 3, 2016

These notes are written to review Fourier series and the other material included in Worksheet 11. Please check the lecture notes for further details.

Quick Review – Section 10.1

The following is a two-point boundary value problem, consisting of a differential equation of order 2 together with suitable boundary conditions.

$$y'' + p(x)y' + q(x)y = g(x), \quad y(\alpha) = y_0, \quad y(\beta) = y_1 \tag{1}$$

Note that this problem has the solution $y = 0$ for all x, regardless of the coefficients $p(x)$ and $q(x)$. We are usually interested in nonzero solutions.

Steps for the solution:

- Find the general solution of the differential equation which is given by (1) by using the methods introduced in Chapter 3.
- Consider the boundary conditions in order to derive the coefficients of the general solution.

Problem.1: Solve the boundary value problem

$$y'' + 4y = \sin x, \; y(0) = 0, \; y(\pi) = 0.$$

In Problem 1, $g(x) = \sin(x)$ is a simple function that we already how to deal with it.

Question: How can we solve the equation given by (1) if $g(x)$ is a more complicated function such as a periodic (piecewise) continuous function (For details on piecewise continuous functions please see page 608)

Fourier Series – Section 10.2

Definition: We can solve many important problems involving partial differential equations, provided that we can express a given function as an infinite sum of sines and/or cosines. These trigonometric series are called **Fourier series**.

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\big(a_m \cos(\frac{m\pi x}{\text{(L)}}) + b_m \sin(\frac{m\pi x}{\text{(L)}})\big). \qquad (\bigstar)$$

↘ What is L?

To discuss Fourier series, we need following tools:

- Periodic functions, in particular periodicity of the Sine and Cosine functions
- Orthogonality of the Sine and Cosine Functions
- Even and odd functions and their integrals

Periodic Functions: A function f is said to be **periodic** with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if

$$f(x+T) = f(x) \tag{2}$$

for every value of x. The smallest value of T for which (2) holds is called the **fundamental period** of f.

Problem 2: Prove that if f and g are periodic functions with common period T, then any linear combination $c_1 f + c_2 g$ is also periodic with period T.

Orthogonality of Sine and Cosine functions: We generalize the concept of orthogonality of vectors. The standard **inner product** (u, v) of two real-valued functions u and v on the interval $\alpha \leq x \leq \beta$ is defined by

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)dx.$$

The functions u and v are said to be **orthogonal** on $\alpha \leq x \leq \beta$ if $(u, v) = 0$.
A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

Problem 2: Prove that the functions $\sin(\frac{m\pi x}{L})$ and $\cos(\frac{m\pi x}{L})$, $m = 1, 2, \ldots$ form a mutually orthogonal set on the interval $-L \leq x \leq L$ and they satisfy the following orthogonality relations: (See page 598 for details)

$$\int_{-L}^{L} \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi x}{L})dx = \begin{cases} 0, & m \neq n \\ L, & m = n; \end{cases}$$

$$\int_{-L}^{L} \cos(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L})dx = 0, \text{ all } m, n;$$

$$\int_{-L}^{L} \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L})dx = \begin{cases} 0, & m \neq n \\ L, & m = n. \end{cases}$$

The Euler-Fourier Formulas: Assume that series of the form ($\bigstar$) converges and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \big(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L})\big)$$

Given such a function $f(x)$ of period $2L$ we can determine the coefficients a_n and b_n (See page 599 for details).

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L}), \quad n = 0, 1, 2, \ldots$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L}), \quad n = 0, 1, 2, \ldots$$

Applications of Fourier Series

Besides their association with the method of separation of variables and partial differential equations, Fourier series are also useful in various other ways, such as in the analysis of mechanical or electrical systems acted on by periodic external forces.
Some of the most important applications can be given as

- **Signal Processing:** Fourier series is used for periodic, continuous signals.You can express any waveform as a sum of sine waves, using Fourier series. Advanced noise cancellation and cell phone network technology is ensured by Fourier Series. It is also used to minimize noise bandwidth and demands respectively.

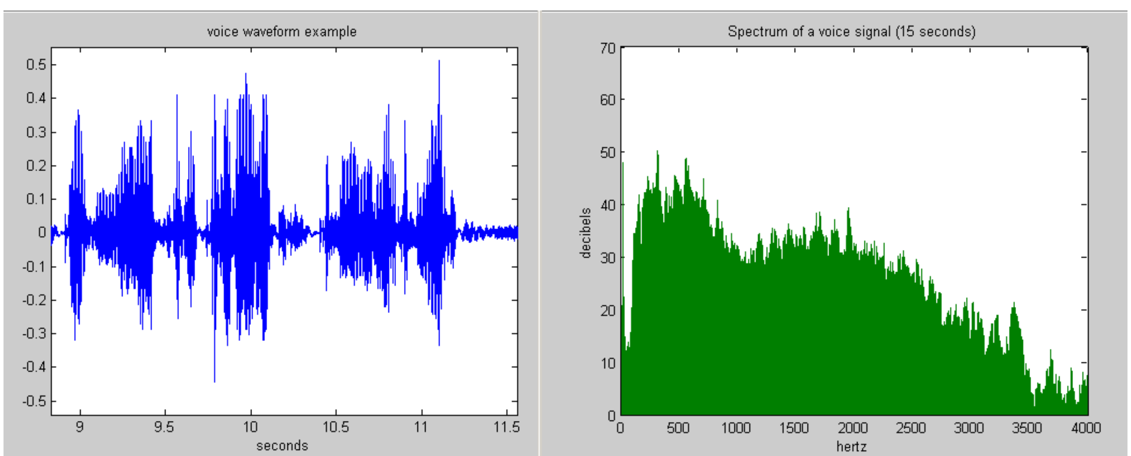

Figure 1: Audio signal in time domain (left) and how it looks like in frequency domain (a.k.a frequency spectrum) after applying Fourier transformation (right) and plotting the frequency components of the resulting series. By looking at the frequency spectrum, it is very easy to tell female and male voices apart as they vary in pitch.

- **Digital Compression:** For example MP3 Compression. An audio track is transformed by using Fourier transformation in the frequency domain. The human ear is able to perceive only a finite range of frequencies. Hence, after converting audio track into the frequency domain, all the audio components with a frequency that sits outside the range are removed. The result is a thinner audio file that sounds almost identical to the original track to the human ear.

- **Approximation Theory:** Fourier series are used to write a function as a trigonometric polynomial.
- **Control Theory:** The Fourier series of functions in the differential equation often gives some prediction about the behavior of the solution of differential equation. They are useful to find out the dynamics of the solution.
- **Partial Differential equation:** We use it to solve higher order partial differential equations by the method of separation of variables.