Newton Polytopes in Algebraic Combinatorics

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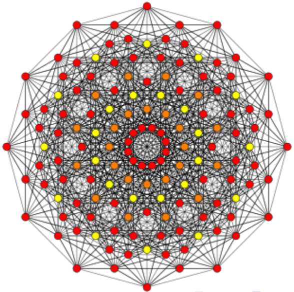
April 6, 2018



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Newton Polytopes

A polytope is a subset of $\mathbb{R}^d$, $d \geq 1$ that is a convex hull of finite set of points in $\mathbb{R}^d$.
Convex polytopes are very useful for analyzing and solving polynomial equations. The interplay between polytopes and polynomials can be traced back to the work of Isaac Newton on plane curve singularities.
The Newton polytope of a polynomial $f = \sum_{\alpha \in \mathbb{Z}^n \geq 0} c_\alpha x^\alpha$ $\in \mathbb{C}[x_1, ..., x_n]$ is the convex hull of its exponent vectors, i.e.,

$$Newton(f) = conv(\{\alpha : c_\alpha \neq 0\}) \subseteq \mathbb{R}^n.$$

Geometric operations on Polytopes

Let P and Q be two polytopes in $\mathbb{R}^d$, then their Minkowski sum is given as

$$P + Q = \{p + q : p \in P,\ q \in Q\}.$$

The geometric operation of taking Minkowski sum of polytopes mirrors the algebraic operation of multiplying polynomials.

Properties:

- $conv(\sum_{i=1}^{n} S_i) = \sum_{i=1}^{n} conv(S_i)$, for $S_i \subseteq \mathbb{R}^n$.
- $Newton(fg) = Newton(f) + Newton(g)$, for $f, g \in \mathbb{C}[x_1, ..., x_n]$.
- $Newton(f + g) = conv(Newton(f) \cup Newton(g))$.

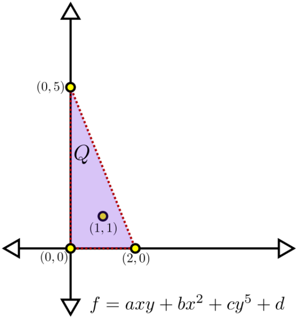
Saturated Newton Polytope–SNP

A polynomial has saturated Newton polytope (SNP) if every lattice point of the convex hull of its exponent vectors corresponds to a monomial.

Instances of SNP in algebraic combinatorics are compiled in the paper "Newton Polytopes in Algebraic Combinatorics" (joint work with Cara Monical and Alexander Yong).

Saturated Newton Polytopes

Example: $f(x, y) = axy + bx^2 + cy^5 + d$ does not have SNP.



Generally, polynomials are not SNP. Worse still, SNP is not preserved by basic polynomial operations.

Example: $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2x_3 + x_2x_4 + x_3x_4$ is SNP but f^2 is not SNP (it misses $x_1x_2x_3x_4$).

The ring of symmetric polynomials

Consider the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in n independent variables $x_1, \ldots, x_n$ with integer coefficients. The symmetric polynomials form a sub-ring

$$Sym_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}.$$

Sym_n is a graded ring: we have

$$Sym_n = \oplus_{k \geq 0} Sym_n^k$$

where Sym_n^k consist of homogeneous polynomials of degree k with zero polynomial. Let $m \geq n$, setting $x_i = 0$ for $i \geq n+1$ defines a surjective homomorphism

$$Sym_m \twoheadrightarrow Sym_n.$$

Let Sym denote the $\varprojlim Sym_n$, i.e., the ring of symmetric functions in countably many variables.

SNP of symmetric functions

Definition

A symmetric polynomial f has SNP if $f(x_1, \ldots, x_m)$ has SNP for all $m \geq 1$.

Q: Do we need check for all $m \geq 1$?

- If $f(x_1, \ldots, x_m)$ has SNP, then $f(x_1, \ldots, x_n)$ has SNP for any $n \leq m$.
- If $f(x_1, \ldots, x_m)$ for $m \geq deg(f)$ has SNP, then $f(x_1, \ldots, x_n)$ has SNP for all $n \geq m$.

Proposition (Stability of SNP)

Suppose $f \in Sym$ has finite degree. Then f has SNP if there exists $m \geq deg(f)$ such that $f(x_1, \ldots, x_m)$ has SNP.

Partitions–Dominance Order

A partition is any sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ of non-negative integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \ldots$$

The length of λ, denoted by ℓ_λ, is the number of parts of the sequence.

The weight of λ, denoted by $|\lambda|$, is the sum of the parts, i.e., $|\lambda| = \lambda_1 + \lambda_2 + \ldots$

$Par(d) = \{\lambda : |\lambda| = d\}$ denotes the set of all partitions of d.

Dominance order $\leq_D$ on $Par(d)$ is defined by

$$\mu \leq_D \lambda \text{ if } \sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i \text{ for all } k \geq 1.$$

Newton polytope of monomial symmetric polynomial

For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

$$m_\lambda(x_1, \ldots, x_n) = \sum_\alpha x^\alpha$$

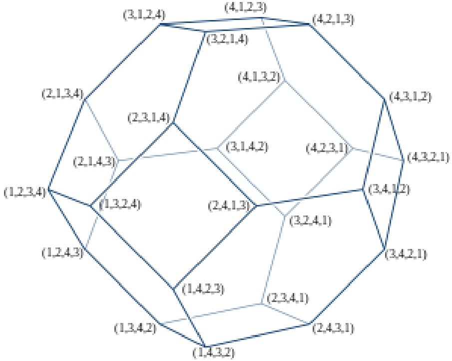
where the sum is over distinct permutations of λ.
The Newton polytope of m_λ is the λ-permutahedron, denoted P_λ, is the convex hull of the S_n orbit of $\lambda \in \mathbb{R}^n$.

$$P_\mu \subseteq P_\lambda \iff \mu \leq_D \lambda.$$

★ $m_\lambda \in Sym$ is SNP $\iff \lambda = (1^n)$.

λ-permutahedron

Let $\lambda = (4, 3, 2, 1)$, then Newton polytope of m_λ is the permutahedron of order 4.



Schur polynomials

We can identify a partition with its Young diagram. A semi-standard Young tableau T is a filling of λ with entries from $\mathbb{Z}_{\geq}0$ that is weakly increasing along rows and strictly increasing down columns. Let $\lambda = (3, 1)$ and the content $\mu = (1, 1, 2, 1)$, then we can have 3 different corresponding tableau:

1	3	3
2	4	

1	2	3
3	4	

1	2	4
3	3	

Then the corresponding Kostka number $K_{\lambda,\mu} = 3$.

- $K_{\lambda,\mu} \neq 0 \iff \mu \leq_D \lambda$

Schur polynomials and SNP

The Schur polynomial is

$$S_\lambda = \sum_{\mu} K_{\lambda,\mu} m_\mu.$$

$$\begin{aligned} Newton(f) &= conv(\cup_{\mu \leq_D \lambda} Newton(m_\mu)) \\ &= conv(\cup_{\mu \leq_D \lambda} P_\mu) \\ &= P_\lambda. \end{aligned}$$

Since $K_{\lambda,\mu} \neq 0$, S_λ has SNP.

Proposition

Suppose $f \in Sym_n$ is homogeneous of degree d such that

$$f = \sum_{\mu \in Par(d)} c_\mu S_\mu.$$

Suppose there exist λ with $c_\lambda \neq 0$ and $c_\mu \neq 0$ only if $\mu \leq_D \lambda$. If $n < \ell_\lambda, f = 0$. Otherwise,

1. *$Newton(f) = P_\lambda \subseteq \mathbb{R}^n$.*
2. *If moreover $c_\mu \geq 0$ for all μ, then f has SNP.*

Schur positivity does not imply SNP

Example (Schur positive without a unique $\leq_D$-maximal term)

Let $f = S_{(8,2,2)} + S_{(6,6)}$. It is enough to show that $f(x_1, x_2, x_3)$ is not SNP. Now, $m_{(8,2,2)}(x_1, x_2, x_3)$ and $m_{(6,6)}(x_1, x_2, x_3)$ appear in the monomial expansion of $f(x_1, x_2, x_3)$. However, $m_{(7,4,1)}(x_1, x_2, x_3)$ is not in $f(x_1, x_2, x_3)$ since $(7, 4, 1)$ is not $\leq_D$ comparable with $(8, 2, 2)$ nor $(6, 6, 0)$.
$(7, 4, 1) = \frac{1}{2}(8, 2, 2) + \frac{1}{2}(6, 6, 0) \in Newton(f)$, then f is not SNP.

Schur positivity SNP

The Schur positivity assumption in the previous Proposition is necessary.

Example

Let $f = S_{(3,1)}(x_1, x_2) - S_{(2,2)}(x_1, x_2) = x_1^3 x_2 + x_1 x_2^3$ is not SNP. It is missig $x_1^2 x_2^2$.

Having a unique $\leq_D$-maximal term is not necessary.

Example

$f = S_{(2,2,2)} + S_{(3,1,1,1)}$ is SNP, but $(2, 2, 2)$ and $(3, 1, 1, 1)$ are $\leq_D$ incomparable.

Products of Schur polynomials are SNP

Proposition

$S_{\lambda^1} S_{\lambda^2} \dots S_{\lambda^N}$ has SNP for any partitions $\lambda^1, \lambda^2, \dots, \lambda^N$.

Proof.

We have

$$S_\lambda S_\mu = \sum_v c^v_{\lambda,\mu} S_v \in Sym$$

where $c^v_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$ is the Littlewood-Richardson coefficient. (We will omit the details here). By homogeneity, $c^v_{\lambda,\mu} = 0$ unless $|v| = |\lambda| + |\mu|$. It can be shown that for $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$. $S_{\lambda+\mu}$ is the unique $\leq_D$ maximal term in the Schur expansion of $S_\lambda S_\mu$; therefore, the product is SNP. □

Involutive automorphism and SNP

Let $w : Sym \to Sym$ be the involutive automorphism defined by

$$w(S_\lambda) = S_{\lambda'},$$

where λ' is the shape obtained by transposing the Young diagram λ.

Example (w does not preserve SNP)

We showed that $f = S_{(8,2,2)} + S_{(6,6)}$ is not SNP. Now

$$w(f) = S_{(3,3,1,1,1,1,1,1)} + S_{(2,2,2,2,2,2)} \in Sym.$$

To see that $w(f)$ has SNP, it suffices to show that any partition v that is a linear combination of rearrangements of $\lambda = (3,3,1,1,1,1,1,1)$ and $\mu = (2,2,2,2,2,2)$ satisfies $v \leq_D \lambda$ or $v \leq_D \mu$.

Forgotten symmetric polynomial

The forgotten symmetric functions are defined by

$$f_\lambda = \epsilon_\lambda w(m_\lambda)$$

where $\epsilon_\lambda = (-1)^{|\lambda|-\ell(\lambda)}$.

Proposition

$f_\lambda \in Sym$ has SNP if and only if $\lambda = (1^n)$.

Proof.

($\Longleftarrow$:) If $\lambda = (1^n)$, then $m_\lambda = S_{(1^n)}$, and $f_\lambda = S_{(n,0,0,\ldots,0)}$ which is SNP.

($\Longrightarrow$:)

$$f_\lambda = \sum_\mu a_{\lambda\mu} m_\mu, \;\; a_{\lambda\mu} \geq 0.$$

It can be shown that if $\lambda \neq (1^n)$, then $a_{\lambda 1^n} = 0$. Also $(1^n) \in P_\mu$ for all $\mu \vdash n$. Then, $(1^n) \in Newton(f_\lambda)$. If $\lambda \neq (1^n)$, f does not have SNP.

Elementary and complete homogeneous symmetric polynomials

The elementary symmetric polynomial is defined by

$$e_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_n}$$

and $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$
The complete symmetric polynomial $h_k(x_1, \ldots, x_n)$ is the sum of all degree k homogeneous polynomials and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$.
Since $e_k = S_{(1^k)}$ and $h_k = S_{(k)}$, then e_λ and h_λ can be considered as product of Schur functions. Therefore, they have SNP.

e-positivity does not imply SNP

$f \in Sym$ is e-positive if $f = \sum_\lambda a_\lambda e_\lambda$ where $a_\lambda \geq 0$ for every λ.
Since $e_\lambda = \sum_\mu K_{\mu',\lambda} S_\mu$, e-positivity implies Schur-positivity.

Example

Look at $f = e_{(3,3,1,1,1,1,1,1)} + e_{(2,2,2,2,2,2)} \in Sym$. In the monomial expansion $m_{(8,2,2)}$ and $m_{(6,6)}$ appear. However, $m_{(7,4,2)}$ does not appear.

Let

$$p_k = \sum_{i=1}^{n} {x_i}^k$$

be the power sum symmetric polynomial. Moreover, let

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_n}.$$

Clearly, p_k does not SNP if $k > 1$ and $n > 1$. Also, p_λ is not SNP for $n > 1$ whenever $\lambda_i \geq 2$ for all i. This is since $x_1^{|\lambda|}$ and $x_2^{|\lambda|}$ both appear as monomials, but $x_1^{|\lambda|-1} x_2$ does not.

Power sum symmetric polynomials and SNP

Proposition

$p_\lambda \in Sym_n$ for $n > \ell(\lambda)$ is SNP if and only if $\lambda = (1^k)$.

Proof.

($\Longleftarrow$:) If $\lambda = (1^k)$, $p_\lambda = e_\lambda$, which is SNP.
($\Longrightarrow$:) Suppose $\lambda_i \geq 2$ and $\ell = \ell_\lambda$. Then since $n > \ell$, then $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_\ell^{\lambda_\ell}$ and $x_2^{\lambda_2} \dots x_\ell^{\lambda_\ell} x_{\ell+1}^{\lambda_1}$ are monomials in p_λ. Thus,

$$(\lambda_1 - 1, \lambda_2, \dots, \lambda_\ell, 1) = \frac{\lambda_1 - 1}{\lambda_1}(\lambda_1, \lambda_2, \dots, \lambda_\ell, 0) + \tag{1}$$

$$\frac{1}{\lambda_1}(0, \lambda_2, \dots, \lambda_\ell, \lambda_1) \in Newton(p_\lambda). \tag{2}$$

However, this can not be an exponent vector since it has $\ell + 1$ nonzero components whereas every monomial of p_λ uses at most ℓ distinct variables. □