Relative ranks of binary forms

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Joint Mathematics Meetings
Atlanta

January 6, 2017

Outline

- Background on the length of binary forms
- Cabinets of binary forms
- Quick review on apolarity
- Sylvester's 1851 Theorem and extension over different fields
- Main theorem which shows that three-length phenomenon occurs in all degrees $d \geq 5$.
- A lower bound for the Waring rank of binary forms
- An improved lower bound for the real Waring rank of binary forms

K-rank of binary forms

Assume that f is a nonzero binary form of degree d with coefficients in a field $K \subseteq \mathbb{C}$. The K-rank of f, $L_K(f)$, is the smallest r for which there exist $\lambda_i, \alpha_i, \beta_i \in K$ such that

$$f(x, y) = \sum_{i=1}^{r} \lambda_i(\alpha_i x + \beta_i y)^d.$$

In case $K = \mathbb{C}$, the K-rank is commonly called the *Waring rank*, and for $K = \mathbb{R}$, it is called as *real Waring rank*.

Some properties of the rank of binary forms

- The following relation is immediate:

$$K_1 \subseteq K_2 \Longrightarrow L_{K_1}(f) \geq L_{K_2}(f) \quad (1)$$

- If g is obtained from f by an invertible linear change of variables over K, then $L_K(f) = L_K(g)$.
- If $f \in K[x, y]$, then $L_K(f) \leq deg(f)$ [*Reznick*].

Sylvester's 1851 Theorem

Theorem

Suppose $f(x,y) = \sum_{j=0}^{d} \binom{d}{j} b_j x^{d-j} y^j \in K[x,y]$ *and suppose* $r \leq d, \alpha_j, \beta_j \in K$ *and* $h(x,y) = \sum_{t=0}^{r} c_t x^{r-t} y^t = \prod_{j=1}^{r} (-\beta_j x + \alpha_j y)$ *is a product of pairwise distinct linear factors. Then there exist* $\lambda_j \in K$ *so that* $f(x,y) = \sum_{j=1}^{r} \lambda_j (\alpha_j x + \beta_j y)^d$ *if and only if*

$$\begin{pmatrix} b_0 & b_1 & \cdots\cdots & b_r \\ b_1 & b_2 & \cdots\cdots & b_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{d-r} & b_{d-r+1} & \cdots\cdots & b_d \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (2)$$

Sylvester form

If (f, h) satisfy the criterion of the Sylvester's Theorem, we shall say that h is a *Sylvester form* for f.

Theorem

Suppose $h(x, y)$ is a Sylvester form of degree r for $f(x, y)$. If S is a splitting field of h, then $L_S(f) \leq r$. If furthermore there is no Sylvester form of degree $r - 1$, then $L_S(f) = r$.

Binary forms with multiple ranks

This following example shows that having multiple ranks over different fields is possible.

Example

Suppose $d \geq 3$, $f(x, y) = (x + \sqrt{5}\ y)^d + (x - \sqrt{5}\ y)^d \in \mathbb{Q}[x, y]$.
Then $L_K(f) = 2$ (if $\sqrt{5} \in K$) and d otherwise.

Question: How many different ranks are possible for a general binary form? (It is a challenging open question).

The following result gives a relation between the degree of a form and the number of possible different ranks:

★ If f has k different ranks, then $deg(f) \geq 2k - 1$ [Reznick].

Binary forms with 3 different ranks

From the previous result, it is possible to have a form f of degree $d \geq 5$ with 3 different ranks.

In the literature there was only one example with three different ranks which was found by Reznick:

Let $f(x,y) = 3x^5 - 20x^3y^2 + 10xy^4$, then $L_K(f) = 3$ if and only if $\sqrt{-1} \in K$. $L_K(f) = 4$ for $K = \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-6})$ (at least) and $L_{\mathbb{R}}(f) = 5$.

Binary forms of full rank

Let f be a binary form of degree d and $L_K(f) = d$, we say that f has *full K-rank*.

Theorem (Blekherman,Sinn)

Suppose f is a real binary form of degree $d \geq 3$ and not a d-th power. Then $L_{\mathbb{R}}(f) = d$ if and only if f is hyperbolic.

Theorem (Białynicki-Birula, Schinzel)

If $d \geq 3$, then $L_{\mathbb{C}}(f) = d$ if and only if there are two distinct linear forms ℓ_0 and ℓ_1 so that $f = {\ell_0}^{d-1}\ell_1$.

Main theorem

Theorem (Reznick, Tokcan)

If $d \geq 5$, then there exists a binary form p_d of degree d which takes at least three different ranks.

Proof.

We prove the theorem by examples.

- $p_{2k}(x,y) = x^k y^k, \; k \geq 3$ for the even degrees.

  $L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k}) = k+1$, $L_{\mathbb{Q}(\zeta_k)}(p_{2k}) = k+2$, and $L_{\mathbb{R}}(p_{2k}) = 2k$.

- $p_{2k+1}(x,y) = x^k y^k (x-y), \; k \geq 2$ for the odd degrees.

  $L_{\mathbb{Q}(\zeta_{k+2})}(p_{2k+1}) = k+1$, $L_{\mathbb{Q}(\zeta_{k+1}})(p_{2k+1}) = k+2$, and $L_{\mathbb{R}}(p_{2k+1}) = 2k+1$.

□

Open Questions

- No binary form with 4 different ranks has shown up in the literature yet. It is a challenging question to determine whether four ranks are possible and it involves solving Diophantine equations.
- How many different ranks are possible for a general binary form?

A lower bound for the Waring rank

We give a lower bound for the Waring rank of binary forms based on their factorization over $\mathbb{C}$ [On the Waring rank of binary forms]

Theorem

Let $m_0 \geq m_1 \geq \ldots m_r$, and suppose that

$$f(x,y) = \prod_{i=0}^{r} \ell_i(x,y)^{m_i} \tag{3}$$

where $r \geq 1$ and ℓ_i's are distinct linear forms. Then $L_{\mathbb{C}}(f) \geq m_0 + 1$.

Proof.

We use the fact that rank is invariant under invertible linear change of variables. After a linear change of variables we may assume $\ell_0 = y$, then we have

$$\tilde{f}(x,y) = y^{m_0} g(x,y) \text{ such that } y \nmid g(x,y). \tag{4}$$

The first m_0 coefficients of $\tilde{f}$ are zero, i.e. $a_0 = \ldots = a_{m_0-1} = 0$ and $a_{m_0} \neq 0$. Note that $deg(\tilde{f}) \geq m_0 + 1$, so by setting $r = m_0$, (3) becomes:

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & a_{m_0} \\ 0 & 0 & \ldots & a_{m_0} & a_{m_0+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ \vdots \\ c_{m_0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence, $a_{m_0} c_{m_0} = a_{m_0} c_{m_0-1} + a_{m_0+1} c_{m_0} = 0$. It follows that $c_{m_0-1} = c_{m_0} = 0$ and every apolar form of degree m_0 is divisible by x^2; therefore, $L_{\mathbb{C}}(f) \geq m_0 + 1$.. □

Improved lower bound for real Waring rank

- Extending the work of Sylvester, Reznick showed that if $f(x, y)$ is a real binary form of degree d, not a d-th power, with τ real roots (counting multiplicities), then $L_{\mathbb{R}}(f) \geq \tau$.

The following theorem improves this lower bound.

Theorem

Let $m_0 = \max(m_0, \ldots, m_r)$ and $n_0 = \max(n_0, \ldots, n_s)$. Suppose that f is a binary form of degree d and not a d-th power with the factorization

$$f(x, y) = \prod_{i=0}^{r} \ell_i(x, y)^{m_i} \prod_{k=0}^{s} p_k(x, y)^{n_k} \tag{5}$$

where ℓ_i's are distinct real linear forms and p_k's are irreducible quadratic forms. Then $L_{\mathbb{R}}(f) \geq \max(\sum_{i=0}^{r} m_i, \max(m_0, n_0) + 1)$.

More details can be found in

- B. Reznick and N. Tokcan, *Binary forms with three different relative ranks*, 2016.
- N. Tokcan, *On the Waring rank of binary forms*, 2016.
- Hopefully in my thesis (Coming soon !) .

THANK YOU!